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A note on the p-elastica in a constant sectional curvature manifold

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Abstract

In this paper, we study the p-elastica, the critical point of the total polynomial curvature functional on those immersed curves satisfying suitable boundary conditions in a Riemannian manifold with constant sectional curvature. We express the torsion of the p-elastica in terms of its curvature in a closed form and completely solve the Euler–Lagrange equation by quadratures. We study the Frenet equation of the p-elastica by using the Killing field.

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1. Introduction

The mathematical theory of elastic rods has been studied for over two centuries. This could trace back to Daniel Bernoulli and Euler in the 1730s [3]. One can study a bent thin rod and consider the energy it stores. The classical Euler–Bernoulli model assigns a numerical value to this energy which is proportional to $\int_0^L k^2(s)\,\mathrm{d}s$. The elastica is the critical point for this total squared curvature functional on regular curves with given boundary conditions. During recent two decades, the Euler–Bernoulli model has been reconsidered for numerous reasons [6,10,12]. The total squared curvature functional has emerged as a useful quantity in the study of geodesics and the closed thin elastic rod is often used as a model for the DNA molecule [11]. Langer and Singer started the research in a series of papers dealing

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with closed elastic curves in spaces of constant sectional curvature and furthermore study the convergence of the negative gradient flow of the total squared curvature [8].

This paper deals with the p-elastic curve, in a manifold with constant sectional curvature, which is a critical point of the total polynomial curvature functional on those immersed curves satisfying suitable boundary conditions. We express the torsion of the p-elastica in terms of its curvature in a closed form and completely solve the Euler–Lagrange equation by quadratures. We find two Killing fields introduced by Langer–Singer [7,9] for the purpose of integrating the structural equations of the p-elastic curves and express the p-elastica in R^3 by quadratures in a system of cylindrical coordinates.

2. Equilibrium equations

Let M be an n-dimensional smooth Riemannian manifold with constant sectional curvature G. The Riemannian metric will be denoted by $\langle\ ,\ \rangle$ and the Riemannian connection by ∇. We have the structural equations:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \tag{2.1}$$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X, Y)Z. \tag{2.2}$$

For vector fields X, Y, Z on M.

Let $\gamma = \gamma(t) : I \to M$ be an immersed curve on M. $T = T(t)$ will denote the unit tangent vector, and v the speed $v(t) = \|\gamma'(t)\| = \langle \gamma'(t), \gamma'(t)\rangle^{1/2}$. The curvature of γ is defined by $k(t) = \|\nabla_T T\|$.

The letter γ will also denote a variation $\gamma = \gamma(w, t) : (-\varepsilon, \varepsilon) \times I \to M$ with $\gamma(0, t) = \gamma(t)$. Associated with such a variation is the variation vector field $W = W(t) = (\partial\gamma/\partial w)(0, t)$ along the curve $\gamma(t)$. We will also write $W = W(w, t)$, $T = T(w, t)$, $v = v(w, t)$, etc., with the obvious meaning. Let s denote the arclength parameter, and we write $\gamma(s)$, $k(w, s)$, etc., for the corresponding reparametrizations. L be the arclength of γ. We may assume $t = s$ be the arclength parameter of γ and then $I = [0, L]$. By a direct computation, We have the following lemma [6].

Lemma 1. *Using the above notation, we have the following formulas*:

1. $[\gamma'(t), W(t)] = 0,$
2. $W(v) = -gv$, *where* $g = -\langle \nabla_T W, T\rangle,$
3. $[W, T] = gT,$
4. $W(\kappa^2) = 2\langle \nabla_T \nabla_T W, \nabla_T T\rangle + 4gk^2 + 2\langle R(W, T)T, \nabla_T T\rangle.$

We consider the energy functional defined on a class of regular curves in M.

$$\int_0^{L(w)} p(k)\, \mathrm{d}s. \tag{2.3}$$

Here $p(k)$ is a polynomial of k with degree ≥ 2 and its leading coefficient is positive. When we confine on the curve $\gamma(t) = \gamma(0, t)$, since $s = t$, we will drop t and s.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}\int_0^{L(w)} p(k)\,\mathrm{d}s\bigg|_{w=0} &= \frac{d}{\mathrm{d}w}\int_I p(k)v\,\mathrm{d}t\bigg|_{w=0} = \int_I \left[p'(k)W(k)v + p(k)\frac{\partial v}{\partial w}\right]\mathrm{d}t\bigg|_{w=0} \\ &= \int_0^L \left[\left\langle R(W,T)T + \nabla_T\nabla_T W, \frac{p'(k)}{k}\nabla_T T\right\rangle + (2kp'(k) - p(k))g\right]\mathrm{d}s. \qquad (2.4)\end{aligned}$$

Here $L(w)$ is the arclength of $\gamma_w(t) = \gamma(w, t)$. We give $\gamma(w, t)$ a boundary condition such that $W(0, 0) = W(0, L) = 0$, $\nabla_T W(0, 0) = \nabla_T W(0, L) = 0$. Then we obtain the first variational formula:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}\int_0^{L(w)} p(k)\,\mathrm{d}s|_{w=0} &= \int_0^L \left\langle \nabla_T^2\left(\frac{p'(k)}{k}\nabla_T T\right) + \frac{p'(k)}{k}G\nabla_T T + \nabla_T[(2kp'(k) - p(k))T], W\right\rangle \mathrm{d}s. \qquad (2.5)\end{aligned}$$

Here we use M being a manifold with constant sectional curvature G. So $R(X, Y)Z = G(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$. We obtain the Euler–Lagrange equation:

$$E = \nabla_T^2\left(\frac{p'(k)}{k}\nabla_T T\right) + \frac{p'(k)}{k}G\nabla_T T + \nabla_T[(2kp'(k) - p(k))T] = 0. \qquad (2.6)$$

Definition 1. A regular unit-speed curve is called a p-elastica if it satisfies the above Euler–Lagrange equation (2.6).

Barros and Garay [2] worked out some similar results in the case $p(k) = (k^2+2)^2$ and applied the critical points of $\int_\gamma (k^2+2)^2\,\mathrm{d}s$ to provide construction methods of Willmore–Chen submanifolds in S^7. When I revise this paper, the referee mentioned a latest paper about closed generalized elastic curves in $S^2(1)$ [1]. In our case, there is a torsion term in the Euler–Lagrange equation (3.3). It might be more complicated and finally we express the torsion of the p-elastica in terms of its curvature in a closed form (3.5).

3. Integration of the *p*-elastica

Suppose γ is a regular curve in an n-dimensional manifold M with constant sectional curvature G, γ has curvatures $\{k_1 = k > 0, k_2 = \tau > 0, k_3, \ldots, k_{n-1}\}$ and the Frenet frame $\{N_0 = T, N_1 = N, N_3 = B, \ldots, N_{n-1}\}$. Then we have the Frenet equations:

$$\nabla_T N_i = -k_i N_{i-1} + k_{i+1}N_{i+1}, \quad i = 0, 1, \ldots, n-1. \qquad (3.1)$$

Here we define $k_0N_0 = k_nN_n = 0$. Now we give the variational formulas by using these curvatures:

$$E = [p^{(3)}(k)(k')^2 + p''(k)k'' + p'(k)(k^2 - \tau^2 + G) - kp(k)]N + (2p''(k)k'\tau + p'(k)\tau')B + p'(k)\tau k_3 N_3. \quad (3.2)$$

Therefore, we have the Euler–Lagrange equation

$$p^{(3)}(k)(k')^2 + p''(k)k'' + p'(k)(k^2 - \tau^2 + G) - kp(k) = 0,$$
$$2p''(k)k'\tau + p'(k)\tau' = 0, \quad k_i = 0, i > 2. \quad (3.3)$$

This implies that we need only consider two- and three-dimensional manifold M in the constant sectional curvature case.

For k is constant, we know τ is constant too and they satisfy

$$p'(k)(k^2 - \tau^2 + G) - kp(k) = 0 \quad (3.4)$$

from the Euler–Lagrange equation (3.3). At this case, the Frenet equation is a linear system with constant coefficients, we can give the formula directly. Now we assume k is not constant. From the second equation, we know

$$p'(k)^2\tau = c_1. \quad (3.5)$$

Here c_1 is a constant. The integral of the first equation becomes:

$$(p''(k)k')^2 + Gp'(k)^2 + \frac{c_1^2}{p'(k)^2} + (kp'(k) - p(k))^2 = c_2. \quad (3.6)$$

Here c_2 is a constant. Therefore, we can express the curvature $k(s)$ by quadratures

$$\pm \int \sqrt{\frac{p'(k)^2 p''(k)^2}{p'(k)^2(c_2 - Gp'(k)^2 - (kp'(k) - p(k))^2) - c_1^2}}\, \mathrm{d}k = \int \mathrm{d}s. \quad (3.7)$$

Definition 2. Let $\gamma(t)$ be a regular unit-speed curve in M. We call a vector field W Killing along $\gamma(t)$ if it annihilates v, k, τ.

By a direct computation, we have the following lemma.

Lemma 2. *In a three-dimensional manifold M with constant sectional curvature G, we have* $W(\tau^2) = 2\langle (1/k)\nabla_T^3 W - (k_s/k^2)\nabla_T^2 W + ((G/k) + k)\nabla_T W - (k_s/k^2)GW + W(1/k)\nabla_T^2 T + 3g\tau B, \tau B\rangle$.

We set the Killing field along the p-elastica $\gamma(s)$ having the form $W = f_1(s)T(s) + f_2(s)N(s) + f_3(s)B(s)$, then the functions f_1, f_2 and f_3 must satisfy the following equations:

$$f_1' - f_2 k = 0, \qquad f_1 k' + f_2'' + f_2(k^2 - \tau^2 + G) - 2f_3'\tau - f_3\tau' = 0,$$
$$f_1(k^2\tau' - 2kk'\tau) + f_2'(3k\tau' - 2k'\tau) + f_2(-k'\tau' + k(-2G\tau + 2\tau^3 + \tau''))$$
$$+ f_3^{(3)}k - f_3''k' + f_3'(Gk + k^3 + 3k\tau^2) - f_3 k'(G - \tau^2) = 0.$$

From these equations and the Euler–Lagrange equation (3.3), we found that the vector fields $J_\gamma = (p'(k)k - p(k))T + p''(k)k'N + p'(k)\tau B$ and $H_\gamma = p'(k)B$ are Killing along the p-elastica γ. The above equations constitutes a linear system whose solution space is six-dimensional in case $\dim M = 3$ and three-dimensional in case $\dim M = 2$. When M is a simply connected manifold of constant sectional curvature, this dimension agrees with the dimension of the isometry group. Thus a Killing field along a p-elastic curve γ can extends to a Killing field on M. Therefore we have the following theorem.

Theorem 1. *Let M be a simply connected manifold with constant sectional curvature G, and let γ be a p-elastica in M. Then the vector fields $J_\gamma = (p'(k)k-p(k))T+p''(k)k'N+p'(k)\tau B$ and $H_\gamma = p'(k)B$ can extend to Killing fields $\tilde{J}_\gamma$ and $\tilde{H}_\gamma$ on M.*

From this theorem, we have the following corollary about the solution of the flow along H_γ.

Corollary 1. *The p-elastica in a simply connected manifold with constant sectional curvature G gives the congruence solutions of the evolution equation $(\partial\gamma/\partial t) = p'(k)B$, the solutions which evolve by symmetries of the isometry group in M.*

In the case $p(k) = k^2 + \lambda$, here λ is a constant, Hasimoto, by using the transform $\psi = k\exp(i\int_0^s \tau\,\mathrm{d}s)$, prove that the evolution equation $(\partial\gamma/\partial t) = p'(k)B$ is equivalent to the non-linear cubic Schrödinger equation [4]. Therefore, it is a completely integrable system.

For two-dimensional manifold M, we consider the integral curve $\tilde{\gamma}$ of the Killing field $\tilde{J}_\gamma$ near the vertex P_0 of γ, the point of which $k(s)$ has an extremum. That is $k'(s_0) = 0$. We have the following theorem.

Theorem 2. *Let γ be a p-elastica in a two-dimensional manifold M with constant sectional curvature G and $P_0 = \gamma(s_0)$ a vertex of γ. Then J_γ is tangent to γ at P_0, the integral curve $\tilde{\gamma}$ in M of $\tilde{J}_\gamma$ through P_0 has curvature $-p'(k(s_0))G/|p'(k(s_0))k(s_0) - p(k(s_0))|$.*

Proof. We denote $\tilde{k}$ the curvature of $\tilde{\gamma}$ and $\{\tilde{T}, \tilde{N}\}$ the Frenet frame of $\tilde{\gamma}$. Then we have $\tilde{T} = \varepsilon T$, $\tilde{N} = \varepsilon N$, here $\varepsilon = \mathrm{sign}(p'(k)k - p(k))$ along the p-elastica γ and $\tilde{k} = \langle \nabla_{\tilde{T}}\tilde{T}, \tilde{N}\rangle$. At the vertex $\gamma(s_0)$, $\nabla_{\tilde{T}}\tilde{T}$ can be considered as the covariant derivative of $\tilde{T}$ along γ at $\gamma(s_0)$ up to a sign. We know at $\gamma(s_0)$

$$\nabla_{\tilde{T}}\tilde{T} = \varepsilon\nabla_T\left(\frac{(p'(k)k - p(k))T + p''k'N}{\sqrt{(p'(k)k - p(k))^2 + (p''k')^2}}\right) = -\frac{p'(k)G}{|p'(k(s_0))k(s_0) - p(k(s_0))|}\tilde{N}.$$

This means

$$\tilde{k} = -\frac{p'(k)G}{|p'(k(s_0))k(s_0) - p(k(s_0))|}. \qquad \square \tag{3.8}$$

In three-dimensional case, the Killing fields J_γ and H_γ can be used to construct a system of cylindrical coordinates. The Euler–Lagrange equation and its first integral imply that

$\nabla_T J_\gamma = -Gp'(k)N$ and $|J_\gamma|^2 = c_2 - Gp'(k)^2$. We could solve the p-elastica as in [9] or [5]. In the following, we will develop it in a special case of R^3.

In R^3, $\nabla_T J_\gamma = 0$. This means that the Killing field J_γ is a constant vector field and it is a translation field. We can obtain one coordinate field $\partial/\partial z = J_\gamma/|J_\gamma|$. Since $J_\gamma \cdot H_\gamma = p'(k)^2\tau = c_1$, H_γ defines a rotation along z direction. $J_1 = J_\gamma - (1/c_1)|J_\gamma|^2 H_\gamma$ is a rotation field perpendicular to J_γ. Thus for some normalization factor, we have $\partial/\partial\theta = QJ_1$. Then $\partial/\partial r$ is given by a cross product

$$\frac{\partial}{\partial r} = \frac{J_\gamma \times B}{|J_\gamma \times B|}. \tag{3.9}$$

There may be a sign due to the sign of $p'(k)$ for the right-hand oriented coordinate system. In the cylindrical coordinate system (z, r, θ), we can write the unit tangent vector as $T = r_s(\partial/\partial r) + \theta_s(\partial/\partial\theta) + z_s(\partial/\partial z)$. Taking the inner products with the above formulas for $\partial/\partial r$ and $\partial/\partial z$, we can obtain

$$r_s = T \cdot \frac{\partial}{\partial r} = \frac{T, J_\gamma, B}{|J_\gamma \times B|} = \frac{p'(k)p''(k)k'}{\sqrt{c_2 p'(k)^2 - c_1^2}}, \tag{3.10}$$

$$z_s = T \cdot \frac{\partial}{\partial z} = \frac{p'(k)k - p(k)}{|J_\gamma|}. \tag{3.11}$$

Choosing a factor Q such that QJ_1 has the proper length at the maxima of $r(s)$, that is, at the maxima of $k(s)$. The first fundamental form of the standard cylindrical coordinate system is $\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. Then the length of $\partial/\partial\theta$ at such a point $\gamma(s)$ is $r = r(s_0)$, the reciprocal of the curvature k_0 of the circle $r = r(s_0), z = z(s_0)$. At this point, the unit tangent vector T has vertical component $T \cdot J_\gamma/|J_\gamma| = (p'(k)k - p(k))/|J_\gamma|$ and the horizontal component is $p'(k)\tau/|J_\gamma|$.

$$k_0 = \frac{|J_\gamma|}{p'(k)\tau}\left|\nabla_T\left(\frac{\partial/\partial\theta}{|\partial/\partial\theta|}\right)\right| = \frac{|J_\gamma|}{p'(k)\tau|J_1|}|\nabla_T J_1| = \frac{|J_\gamma|^3}{c_1|J_1|}. \tag{3.12}$$

Thus we have $Q = c_1/|J_\gamma|^3 = c_1/\sqrt{c_2^3}$ and

$$\theta_s = \frac{T \cdot (\partial/\partial\theta)}{|\partial/\partial\theta|^2} = \frac{\sqrt{c_2^3}(p'(k)k - p(k))}{c_1|J_1|^2}. \tag{3.13}$$

Therefore we have the following theorem.

Theorem 3. *Let (r, θ, z) be cylindrical coordinates given above, and $\gamma(s) = (r(s), \theta(s), z(s))$. Then we have*

$$r_s = \frac{p'(k)p''(k)k'}{\sqrt{c_2 p'(k)^2 - c_1^2}} \qquad z_s = \frac{p'(k)k - p(k)}{\sqrt{c_2}} \qquad \theta_s = \frac{\sqrt{c_2^3}(p'(k)k - p(k))}{c_1|J_1|^2}. \tag{3.14}$$

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